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Modelling of AGM-style doxastic operations in three-valued setting

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Abstract. The goal of our work is to show how a theoretical approach to modeling of reasoning can be analyzed to identify controversial issues that reveal prospects for further research. We will consider one of the basic approaches to modeling of reasoning based on the concept of belief revision AGM, which is viewed as classical because it formulates the basic concepts of belief, introduces the main ways of representing beliefs, cognitive actions, systems of postulates for cognitive actions and the basic principles for constructing epistemic systems. However, this conceptual foundation raises many controversial issues that require further research, such as the problem of purity of the doxastic operations, the problem of primacy of the doxastic operations and the problem of connection between the doxastic operations. To find a possible solution to these controversial points, we will attempt to model the main ideas of AGM within the framework of standard consistent, and complete logic Ł3. The basic principle of our translation is the scheme for constructing an epistemic theory proposed by Gärdenfors, which is considered the basis of AGM. We use a strict three-valued logic formalism to constrain the functioning of doxastic operators and to test how they will function when trying to express the corresponding AGM postulates in a given system. It will allow us to approach the solution of the classical AGM problems or at least to present them from a different perspective. We consider the fundamental possibility of obtaining other doxastic operators in this way and also show how we can implement the minimality criterion for the contraction operator by combining several theorems of three-valued logic. The presented method of translating an informal conceptual scheme into formal logic is convenient for teaching students the basics of modeling and makes it possible to demonstrate the relationships and limitations of the modeled objects and processes.

Keywords: modeling of reasoning, belief revision, cognitive actions, doxastic operators, AGM, three-valued logic

1. Introduction

Modeling as a scientific method and educational approach is widely used in various fields. One of its applications is modeling of reasoning. It can be carried out in two dimensions: as a modeling of formal reasoning (systems of inferences and proofs) and as a modeling of real reasoning (arguments and beliefs). Schemes of reasoning developed by mankind are means of gaining knowledge, prediction and creativity. Modeling of ordinary reasoning also provides an opportunity to get closer to an understanding of informational influences, which take the form of information warfare nowadays, and to predict possible ways for a person to protect against destructive informational influences.

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Modeling of reasoning at the level of ordinary reasoning. Modeling of reasoning is a fruitful area of scientific research that encompasses various fields in theory and practice. Modeling of reasoning aims at various processes: decision-making, drawing conclusions, linking data in databases, creating information patterns, etc. In teaching practice, we also deal with modeling of reasoning from different angles. On the one hand, it is about the creation of models of argumentation within the framework of the theory of argumentation and critical thinking. On the other hand, we are concerned with the use of reasoning as a basis for considering formal systems. In both cases, we deal with two levels of reasoning – real and formal ones. Both levels have their modeling peculiarities. Real (or ordinary) reasoning is too hard to formalize because it is so complex that it sometimes seems illogical. Formal reasoning seems so abstract that students do not apprehend how to apply it in practice.

There is an almost universally accepted list of characteristics of ordinary reasoning that shows why it is so difficult to model.

"The following set of features are characteristic of much everyday 'logical reasoning', yet formal logic embodies none of them.

1. Reasoning is often context dependent. A deduction that is justifiable under one set of circumstances may be flat wrong in a different situation.
2. Reasoning is not always linear.
3. Reasoning is often holistic.
4. The information on which the reasoning is based is often not known to be true. The reasoner must, as far as possible, ascertain and remember the source of the evidentiary information used and maintain an estimation of its likelihood of being reliable.
5. Reasoning often involves searching for information to support a particular step. This may involve looking deeper at an existing source or searching for an alternative source.
6. Reasoners often have to make decisions based on incomplete information.
7. Reasoners sometimes encounter and must decide between conflicting information.
8. Reasoning often involves the formulation of a hypothesis followed by a search for information that either confirms or denies that hypothesis.
9. Reasoning often requires backtracking and examining your assumptions.
10. Reasoners often make unconscious use of tacit knowledge, which they may be unable to articulate" [3, p. 2].

Various researchers focus on some of the items on this list and try to overcome them step by step by formulating numerous algorithms and methods for modeling ordinary reasoning in more detail to make the model as close to reality as possible. However, modeling of reasoning is carried out not only in different areas but also at different levels. We usually offer students training tasks in which they have to simulate a real process according to formal rules. By performing them, students learn to reconcile real cases with formal requirements. The tasks that demand to analyze the requirements themselves, comparing rule systems, comparing two or more models, rising to a more formal level and evaluating the adequacy of the model are less common.

Modeling of reasoning at the formal level. However, the formal level is essential because it provides the basis for a fundamental understanding of reasoning. The definition of the basic concepts and their relationships, which are incorporated into the modeled system, is carried out precisely at the theoretical level. However, when we give students tasks related to modeling of reasoning, we take the abstract theoretical basis for granted and do not consider it separately. For example, we assume that the students have already learnt what knowledge is and how it differs from belief; they know that knowledge can be represented in terms of propositions on which actions can be taken; we also assume that these actions are obvious – we expand or reduce our knowledge, etc. All these objects and cognitive actions are the subject of a separate research carried out at a more abstract, formal level. Its purpose is to clarify the basic features of the processes of changing beliefs, to develop a common understanding of the concept of a cognitive action and its possible options, to establish connections between types of cognitive actions, etc. However, even formal concepts are not free from shortcomings and inconsistencies; therefore, they require constant revision and completion.

In everyday communication, we operate with information easily. Similarly, we acquire and lose our beliefs, restore them and change them easily. In addition, we are focused on the acquisition of knowledge, so we regard it as an obvious thing that the procedure for expanding knowledge always has priority, but at the same time, we pay very little attention to procedures that implement contraction, rejection of beliefs. Analyzing the relevant rules is also essential.

For example, most epistemic systems support the criterion of priority of new information because it corresponds to real cognitive actions as we always pay attention to new information. The consequence of accepting new information can cause the appearance of a contradiction, so there are many rules for processing, excluding or isolating contradictory propositions. While working with real cases similarly, we face the problem of fake news and misinformation, which may not contradict previous beliefs as they may be either nonsense based on a different conceptual scheme or confirmed in inaccessible ways. Such information does not trigger the mechanisms of standard processing of contradictions but is accepted 'until clarification', over time destroying our worldview scheme by turning it into a pile of inconsistent information that can be neither definitively rejected nor confirmed. The question arises if it is worth reviewing the principle of priority of new information and exploring possible ways of its implementation.

A generally accepted way of checking the adequacy of a theory or a scheme is its interpretation using a real case, but it is no less productive to interpret it using another formal scheme. Backwards interpretation is also a useful task. After formal analysis, we can obtain new formulations of rules and previously unknown consequences of known rules. They need to be interpreted in real reasoning.

Modeling of reasoning and teaching. Dealing with epistemic systems relating to knowledge representation schemes and operations on them is important for learning in several ways.

Firstly, the students are considered as epistemic agents participating directly in the functioning of knowledge. Thus, the modeling of cognitive activity always has at least one case for checking adequacy. Simultaneously, such a check also has the opposite effect as it affects the formation of the appropriate reasoning schemes of the epistemic agent. In other words, if the students work with schemes of reasoning, they not only check them for consistency and correspondence but also capture the correct ones and implements them in their cognitive activity. Modeling of reasoning in the educational process makes it possible to build models of two types: 'model of' and 'model for' [6, p. 52].

Secondly, epistemic logic has many semantic interpretations. For example, modal interpretation and semantics of possible worlds, multi-valued logics, probabilistic interpretation, etc. Different formal interpretations of real cognitive activity serve as good educational examples directly related to human activity but not to cases that the student will not meet anywhere else apart from studying.

Thirdly, epistemic systems well demonstrate the presence of different levels of formalization: from a set of individual cases to the highest level of the abstract criteria of rationality. Simultaneously, semi-formal conditions and informal agreements are also presented in epistemic systems. This fact allows us to consider not only the issue of consistency of the real process and its model but also the issue of consistency of models and schemes of different levels of formalization.

The goal of our work is to show how a theoretical approach to modeling of reasoning can be analyzed to identify controversial issues that reveal prospects for further research. We will consider one of the basic approaches to modeling of reasoning based on the concept of belief revision AGM, which is considered classical because it formulates the basic concepts of belief, introduces the main ways of representing beliefs, cognitive actions, systems of postulates for cognitive actions and the basic principles for constructing epistemic systems. However, this conceptual foundation raises many controversial issues that require further research. To find a possible solution to these controversial points, we will attempt to model the main ideas of AGM within the framework of standard consistent logic. It will allow us at least to present them from a different perspective if not to solve the classical AGM problems. From the viewpoint of teaching modeling methods to students, the approach we present shows a way to apply formal logic to the analysis of complex meaningful concepts. Namely, we consider a method of translating an informal system into a prefabricated axiomatic system to check its consistency. On the other hand, we obtain a new meaningful interpretation of the standard logic.

The plan of our work is as follows.

1. We will consider the epistemic system of AGM, in which the main concepts of belief revision are formulated. This system is semi-formal; the principles of cognitive actions are defined within it as general guidelines not being subject to strict formalism, but the general principles of set theory are used.
2. We will consider the concept of cognitive actions, their types and some problems that concern researchers but do not receive a satisfactory solution.
3. We will formulate the basic principles of interpretation of cognitive actions of AGM as doxastic modal operators of a three-valued logic.
4. We will check what properties the obtained operators exhibit, what regularities and connections exist between them, and how we can express the postulates of AGM cognitive operations in a three-valued logic to obtain theorems. Obviously, we can use tables to validate the resulting expressions.
5. According to the received interpretation, we will review the minimality criterion for its compliance with the AGM postulate and offer options for its formulation.

2. The AGM approach in belief revision

Belief revision is a promising trend in modern epistemic logic. It deals with changes of our knowledge and beliefs, and its goal is to provide a formal account of the process of belief change. This theory has many practical applications in AI systems, decision-making systems, databases, system update procedures and others [2, 4]. Belief revision has also inspired much theoretical research in the fields of logic and cognitive science.

The best-known trend in belief revision is a so-called *AGM approach* named after its initiators Carlos E. Alchourrón, Peter Gärdenfors and David Makinson [1]. This system is considered to be classical. First, it has rather long roots. The initial ideas developed in the framework of AGM date back to the second half of the 20th century (Hansson [7], Levi [10], Segerberg [12], Stalnaker [15]). Then, it is quite fundamental and very productive. Nowadays, AGM is a quite elaborate approach to the analysis of the belief dynamics. This framework has generated the basic concepts and principles for considering beliefs, their systems and the ways in which they can be changed.

The main goal of AGM is to develop general concepts for belief change so that one can describe and create an acceptable model for rational cognitive actions. Similar to any theory, AGM requires a considerably high level of idealization. Most of these idealizations are described in epistemic and doxastic logics (Hintikka [9], von Wright [17]). The idealization presupposes the presence of a rational agent who cares about their knowledge, who is aware of their knowledge, who is ready to revise and order it, who knows the rules of logic and how to deduce consequences. In general, these are quite acceptable and even desirable idealizations.

AGM is designed to be able to apply its principles to the development of models of the belief dynamics, regardless of scope. Moreover, each additional interpretation enriches in some way the understanding of the interpreted subject. The main idea of this work is to see how we can represent cognitive actions considered in AGM within certain logic and to see what interesting properties can be discovered in this way. Then, we consider the basic concepts and principles of AGM. After that, we interpret them in a logical scheme. Finally, we see what information we can obtain about them in the resulting system.

2.1. General rules for the belief dynamics

The AGM system formulates several basic principles that the belief dynamics shall comply with. We can distinguish two types of these principles. The first group concerns a general understanding of the dynamics of beliefs. These principles are not very formal but essential. The second group includes so-called AGM postulates, which directly describe the properties of expansion, contraction, and revision operations in a more formal way. We will call the principles of the first group 'the general rules' and the principles of the second group 'the postulates of doxastic operators'. Basically, the group of general rules describes the field of belief revision. Let us summarize them.

1. All agent's beliefs can and should be represented as propositions. In this case, we need to note that each belief can be represented as a particular formula. Thus, we can use the familiar notation A, $\neg A$, $A \vee B$ to express beliefs and the relations between them in a belief set.

2. A proposition is said to be a belief if an agent accepts it. Therefore, we can divide all sentences in language into two groups: the sentences included in the agent's set of beliefs and the sentences not included in the agent's set of beliefs. It seems obvious that such a situation can be interpreted simply using two-valued logic, in which the truth value of the corresponding proposition is determined by its presence or absence in the set of beliefs.
3. One of the basic rules of belief revision looks fundamental when it is voiced: 'the belief dynamics is possible'. In other words, the agent can change their beliefs. From a viewpoint of logic, it means that nothing else than those propositions can change their truth values after certain actions have been performed.
4. Belief changes should come from ordinary reasoning. Belief revision is an actual area of research whose followers are trying to work out in detail the possible options for belief changes. However, ordinary reasoning is too diverse and complex to be described in a strict formalism. Nevertheless, we must not lose the logic.
5. Belief changes should be rational. The meta-level of epistemological theory contains criteria of rationality used to evaluate the other factors of the theory. Criteria of rationality are used to determine the behaviour of belief changes [5, p. 8].

2.2. Gärdenfors's four components of epistemic theory

We will follow the classical definitions of Gärdenfors [5] and try to implement them. He distinguishes four components of epistemic theory in his fundamental book *Knowledge in flux* [5].

Epistemic states are used to represent a current or possible cognitive state of a rational agent in a certain moment. An epistemic state is "in equilibrium" if it is consistent and satisfies the criteria of rationality.

Epistemic attitudes are the statuses of beliefs included in an epistemic state. For example, in the model based on propositions, the epistemic attitudes may be 'accepted', 'rejected', 'indetermined'.

Epistemic inputs – ways to initiate change. If we assume that the epistemic state is internally stable, then its changes require external stimuli, the so-called 'epistemic inputs'. These inputs cause 'belief changes' and the transformation of the original epistemic state into a new epistemic state.

Criteria of rationality – ways to manage change. Here are the most commonly used criteria: *primacy of new information* (the new information is always accepted); *consistency* (the new epistemic state ought to be consistent if it is possible); *minimal loss of previous beliefs* (the attempt to retain as much of the old beliefs as possible).

Gärdenfors [5] tries to keep the specified parameters as formal as possible to cover any area of application. He writes: "The epistemological theories are conceptualistic in the sense that they do not presume any account of an 'external world' outside of the individuals' epistemic states.

It is true that the epistemic inputs in general have their origin in such a 'reality', but I argue that epistemic states and changes of such states as well as the rationality criteria governing epistemic dynamics can be, and should be, formulated independently of the factual connections between the epistemic inputs and the outer world" [5, p. 9].

2.3. Beliefs and cognitive action

At first, AGM deals with the concepts of belief, a set of beliefs and cognitive actions. A belief in AGM can be considered as a proposition that the agent believes to be true. A belief is usually denoted as **A** (or another capital letter).

Gärdenfors [5] writes that belief states are a representation of a person's knowledge and beliefs at a particular point in time. However, they do not contain psychological entities but are represented as rational idealizations of psychological states. It means that a state in a computer program can also be seen as a model of an epistemic state [5, p. 7].

Set of beliefs **K** is the set of propositions (sentences) that the agent believes to be true. The set of beliefs has the following properties: it is consistent and closed under the relation of logical consequence. Proposition **A** is derivable in set of beliefs **K** if it belongs to the set of beliefs; see Expression (1).

$$K \vdash A \text{ iff } A \in K \tag{1}$$

Gärdenfors [5] says directly that a proposition can have three statuses regarding a belief set.

Accepted: belief A is accepted iff $A \in K$

Rejected: belief A is rejected iff $A \notin K$

Indetermined: else the status of A is indetermined

AGM develops the concept of a cognitive actions representing belief changes. Since we deal with beliefs and all actions will be executed with beliefs, we can use the term 'doxastic' to name our interpretation of cognitive actions. So, now we are talking about doxastic operations, with the help of which we can implement different cognitive actions.

A cognitive action is an operation on a sentence that determines its status with respect to the agent's set of beliefs. Gärdenfors [5] explicitly points out that cognitive actions shall implement scenarios for changing the status of beliefs. He defines only 6 such scenarios: changing the statuses of the belief from indetermined to accepted (1) and from indetermined to rejected (2) are implemented by expansion; from accepted to rejected (3) and from rejected to accepted (4) are implemented by revision; from accepted to indetermined (5) and from rejected to indetermined (6) are implemented by contraction. Thus, he associates a certain doxastic operation with each pairs of scenarios.

The basic doxastic operations are *expansion* and *contraction.*

AGM expansion: $K + A$ (K – an initial belief set, A – a new belief added to it).

AGM contraction: $K \div A$ (K – an initial belief set, A – a belief extracted from it).

The main postulates for these operations have been described in the AGM, so this system is sometimes called the 'postulate approach' [5], [7]. The basic properties of expansion and contraction are presented in the AGM as a set of formal axioms. It is generally accepted that any interpretation of doxastic operations should satisfy the appropriate set of axioms.

Giving its name as a title for a field of research, the revision as a cognitive action is considered a complex action that can be represented as a combination of expansion and contraction. In this paper, we will focus only on expansion and contraction. Nevertheless, many other operations can also be defined by these operations, such as the operation of updating, consolidation and so on. Moreover, we can implement expansion and contraction in different ways. These possibilities depend on the format of the main epistemic objects, such as epistemic states, allowed epistemic inputs, expected epistemic outcomes and so on.

Despite conducted research, there are some interesting issues in the belief dynamics. Some of them are fundamental questions that cannot be solved when being based only on empirical clarifications.

1. *The problem of purity of the doxastic operations.* In other words, it is the question if the expansion not accompanied by contraction is possible. On the other hand, one can ask the same for contraction: whether it is possible to have a contraction not accompanied by a previous or further expansion. There is possibly a more precise question: can there be a contraction that is not caused by an expansion?
2. *The problem of primacy of the doxastic operations.* It is generally accepted that expansion and contraction are the two fundamental doxastic operations from which the revision can be defined. Are they equal in status and can both be considered primary? If we can determine one of these operations to be primary, we can also argue that the belief dynamics is determined by that particular operation.
3. *The problem of connection between the doxastic operations.* The connection between expansion and contraction is not fully understood. They can be either expressed by the other or independent [8].

To investigate such issues, it is more necessary to use intra-theoretical or cross-theoretical research. One of the possible approaches may be to immerse the semi-formal system of AGM into an adequate formal system of complete logic.

3. Doxastic action in a three-valued setting

Following Gärdenfors [5], we distinguish such main components of a belief system as a class of models of epistemic states, a valuation function determining the epistemic attitudes, a class of epistemic inputs and a commitment function that is defined for all epistemic states and all epistemic inputs [5, p. 18]. It would be interesting to consider a possible interpretation of the doxastic operations on a three-valued logic. As the main one, it makes sense to take Lukasiewicz's three-valued logic Ł3. We believe that within the framework of this logic, we can construct a system that meets Gärdenfors's requirement for a belief system.

1. *A class of models of epistemic states* can be represented through a set of propositions.

2. *Valuation function* v(A) determining the epistemic attitudes can also be easily implemented because Ł3 logic considers a third value that approximates our needs.
   A value ½ is the value a proposition if we cannot say anything about its trueness. In our case, this is value *n*, which we interpret as a value of a proposition not being in the agent's belief set. We have to consider three options:
   v(A) = t – the agent believes in A or is confident in A (proposition A is in their belief set);
   v(A) = n – the agent is not confident in A so does not use A as their belief (proposition A is not in their belief set and ¬A is not in their belief set either).
   v(A) = f – the agent believes in ¬A (proposition ¬A is in their belief set).
3. *A class of epistemic inputs and a commitment function* can be implemented by derivability. The derivability of a formula can be interpreted as the realization of some kind of epistemic input. In general, we can take $\vdash A$ (A is derivable) as the evidence that A is true, and in this case, it does not matter how the agent learns about it. In other words, we want to consider $\vdash A$ as the result of an external validation of A (e.g., A is a fact) or as internally derivable (A is a consequence of the agent's beliefs). In any case, the agent's behavior must be the same. Based on the principle of primacy of new information adopted in AGM, the agent shall accept A as their belief.

Moreover, the Ł3 logic not only describes the appropriate domain but is also complete. Thus, we have two tasks.

1. We need to implement the correct interpretation of the doxastic operations that allow us to meet the requirements of AGM.
2. We need to obtain an interpretation of the doxastic AGM operations by the Ł3 connectives.

If we accomplish these tasks within Ł3, we can see the properties of the doxastic operators that can be described in complete logic. We want to use Ł3 as a delimiter for the doxastic operators to get closer to solving some of the fundamental issues of AGM at a higher formal level. Moreover, we have a good algorithm to check the adequacy of the interpreted properties without involving additional semantics. Semantic tables will also show us exactly which propositional values produce the corresponding results.

3.1. The basic notations and expressions

AGM assumes that cognitive actions are binary operations on pair (K, A): a set of beliefs and a proposition. The result of these operations is also a set of beliefs. Since the agent possesses the belief set, and the belief set is always fixed in AGM, we can omit this notation. Thus, we can consider the set of true propositions supported by the agent as a set of beliefs and consider that all operations are carried out with it.

If we need to explicitly show the original set of beliefs, we can label it with a formula of propositional logic, such as B. In more detail, we can represent B as a conjunction of those propositions that the agent believes and their consequences.

The situation, when a certain proposition A belongs to the agent's set of beliefs, can be represented in several ways:
A – proposition A belongs to the original set of beliefs;

A&B – proposition A and (set of propositions, implied) B, belong to the original set of beliefs;
B→A – proposition A belongs to the set of beliefs described by set of propositions B.

Thus, non-doxastic expressions (i.e. which do not contain doxastic operators) A, $\neg A$, B, A&B, A∨B, $\neg A \& B$ represent certain sets of beliefs since each formula can be used as a label for a complex combination of propositions that represent the agent's beliefs. Gärdenfors [5] shows cognitive actions as those that change the status of propositions; this approach gives us a reason to think of them as unary operators dealing with the truth values of a particular proposition. Accordingly, we can represent doxastic operations as modal operators [13].

+A – expansion: proposition A is added to the set of beliefs.

÷A – contraction: proposition A is removed from the set of beliefs.

When we consider other propositions of the set of beliefs not explicitly affected by the doxastic operation, we need to explicitly name the original set of beliefs: +A&B.

In addition, we should carefully consider the interpretation of connectives in the context of their ability to display the statuses of complex propositions in the belief dynamics.

Negation. Initial belief can be true, unknown or false. The ordinary Ł3 negation turns true to false, false to true and leaves an unknown value without changes. Thus, we have two kinds of proposition statuses: *determined* — true (t) or false (f), and *indetermined* — *n*, when the agent knows nothing about the proposition status. If a sentence has a determined status it can be negated by ordinary rules. If it is indetermined, it means that neither this sentence nor its negation belongs to a belief set; naturally, the belief set cannot be negated. In this case, the indetermined value *n* presents the application conditions of the operator. It can be read as follows: if a proposition is indetermined, then its negation is indetermined too, – negation of the unknown is unknown (table 1). Thus, we can assume that the tabular definition of negation adequately reflects the properties we need.

Table 1
Table for negation.

$\neg$	p
f	t
n	n
t	f

Implication. We hope greatly that implication can adequately express the connections both between beliefs and between the propositions not considered beliefs. We can order truth values in this way: $t > n > f$. It makes sense to think that a proposition whose value is currently undefined can acquire the status of true or false after receiving additional information. (Roughly speaking, it will turn out to be *t* or *f* when the agent learns about it). Simultaneously, the true and false values represent the maximum and minimum of the truth values, respectively. So, we can use ordinary implication rules: implication $x \to y$ is true if $y \geqslant x$. We try to follow the ways of natural reasoning. Thus, implication is true in case if: consequent is true (the truth follows from everything); antecedent is false (everything follows from false); uncertainty follows from uncertainty. Implication is unknown in other cases if only one of its components is defined. In the rest, we have only two cases. The one is when the antecedent is true and the consequent is unknown, and the other case occurs when the antecedent is unknown and the consequent is false. The value of the implication in these cases is indetermined. Implication is false if its antecedent is true and its consequent is false (table 2).

Other connectives are entered as follows: $v(p \vee q)$ = max(v(p), v(q)), $v(p \& q)$ = min(v(p), v(q)).

Table 2
Table for implication.

$\rightarrow$	t	n	f
t	t	n	f
n	t	t	n
f	t	t	t

3.2. Expansion and contraction as modal operators

Based on the idea of Shramko [13], we can try to consider doxastic operations as unary operators like modal operators. Thus, we can consider the expansion as unary operator +A, and it means that proposition A is added to the set of beliefs. In other words, the agent, who possesses the beliefs, considers the proposition true. This interpretation completely determines the value of expression +A. The agent does not consider true a proposition that is false or unknown to him. Therefore, expression +A will be true only when A is true.

Gärdenfors [5] claims that there are six possible ways of belief change. Two of them are 1. from indetermined to accepted; 2. from indetermined to rejected. In addition, they define the operation of expansion. At first, it seems that we should represent the expansion operator as one that converts the value of *n* to true or false, but the expansion does not occur that way. The expansion captures that the proposition becomes the agent's belief. Respectively, a belief is a proposition that the agent believes to be true. We need to note the success of the expansion. Thus, the proposition is successfully added if the agent considers it to be true.

Table 3
Table for expansion.

$+$	p
t	t
f	n
f	f

After expansion by A, sentence A can no longer be unknown or indetermined (i.e. +A = f if A = n). Depending on the truth value of A, A or not A can be added to the set of beliefs. Respectively, A becomes accepted or rejected.

If an agent considers the sentence to be true, they add it to their belief set. We obtained +A. If an agent considers the sentence to be false, they add not A to their belief set. We obtained $+\neg A$.

"The first type is when the epistemic attitude 'A is indetermined' is changed into either 'A is accepted' or '—A is accepted' (that is, 'A is rejected'). I call this kind of change an expansion, because it consists in adding a new belief (and its consequences) to the belief set without retracting any of the old beliefs," wrote Gärdenfors [5, p. 47]. Unary expansion operator $+$ successfully completes this task: it returns true only when it succeeds, that is, when the added proposition is considered true.

The tabular definition of the expansion operator corresponds to the formula in Ł3 $\neg(p \to \neg p)$, i.e. it is not true that p is non-true. It is also obvious that the definition of the expansion operator completely coincides with the interpretation of the necessity operator $\Box$ in the system Ł3 (or with an external approval operator $\vdash$). Thus, we can obtain additional information about the expansion operator by comparing it with a normal modal operator.

However, the more interesting fact is that this interpretation of the expansion as a unary operator allows us to describe an indetermined proposition as one that takes the value *n* in the case when neither the proposition A nor its negation $\neg A$ is a belief.

"The second kind of change occurs when one of the attitudes 'A is accepted' or 'A is rejected' is changed into 'A is indetermined'. This kind of change is called a contraction, because it consists in giving up the belief in A (or the belief in — A). This kind of change can be made by an agent in order to open up for investigation some proposition that contradicts what the agent previously believed" [5, p. 47]. Unary contraction operator $\div$ successfully completes this task: it returns true only when it succeeds, that is, when the proposition is not added, and neither A nor $\neg A$ is not believed (expression 2).

$$\neg{+}A \;\&\; \neg{+}\neg A \tag{2}$$

Expression 2 describes a state of the sentence that is unknown to the agent. Accordingly, this expression takes the value of *t* only when proposition A takes the value of *n*. In other cases, when A is known to the agent, the sentence takes the value of *f*. This attribution of true values is fully consistent with the description of the contraction operation $\div$. In AGM, a contraction is an operation that changes the status of a proposition from determined to indetermined. Thus, $\div$ shows actually the removal of the proposition from the agent's belief set.

$$\div A = \neg{+}A \;\&\; \neg{+}\neg A \tag{3}$$

Table 4
Table for contraction.

$\div$	p
f	t
t	n
f	f

With $\div A$ we can formulate the law of the excluded fourth, which describes exhaustively the set of possible states of a sentence with respect to the set of beliefs of the agent. We can prove the corresponding theorem in Ł3 using a truth table.

$$A \vee \div A \vee \neg A \tag{4}$$

Thus, it looks like we have chosen the correct interpretation of the operators. Moreover, it is noteworthy that the doxastic operators introduced in this way demonstrate significantly different properties. +A functions as a normal modal operator and satisfies the corresponding properties. This operator also allows iteration and can be applied to the doxastic formulae. So, we can successfully use expressions like ++A = +A and $+\div A$ = $\div A$.

In turn, operator $\div$ applied to the doxastic formulae returns a contradictory formula. Both expansion and contraction deal with three values t; n; f and map them on two values t; f. Thus, contraction applied to the formula with only t and f values without n gives us only false.

Because + functions as a normal strong modal operator, it is possible to formulate an introduction rule for the expansion operator that allows us to handle the epistemic input and at the same time implement the priority of new information. The rule is well known in modal logic as the rule of necessitation **N** (expression 5).

$$\text{If } \vdash A \text{ then } \vdash +A \tag{5}$$

It would be interesting to formulate an introduction rule for the contraction operator. To do this, we need to take into account the conditions of its implementation and think about what the epistemic input should be whose result of the processing should be a contraction. No matter how banal it may sound, the following rule reflects well the implementation of the epistemic input for contraction.

$$\text{If } \vdash +A \text{ and } \vdash +\neg A \text{ then } \vdash \div A \tag{6}$$

The doxastic expansion and contraction operators defined in this way perform the function of identifying the status of the proposition (table 5).

Table 5
Statuses of proposition.

Action	**Status**	**Expressions**
$+A$	A is exactly true (only true)	$A \in K$, $K \to A$, $K\&A$
$+\neg A$	A is exactly false (only false)	$\neg A \in K$, $K \to \neg A$, $K\&\neg A$
$\div A = \div\neg A$	A is not defined (not true and not false)	$A \notin K$ and $\neg A \notin K$
$\neg + A$	A is not true (A can be indefinite, and can be false)	$A \notin K$

We can also express some links between these operators.

$$\neg + A = \neg A \vee \div A \tag{7}$$

So far, everything seems to be occurring well, and we can prove the corresponding theorems.

4. The AGM postulates of expansion in a three-valued setting

Having identified the key ideas of our interpretation, we approached the main task. We need to consider the AGM postulates defining the expansion and contraction operations and expressing them in a language that we have determined are appropriate for the three-valued logic. We translate the AGM postulates into theorems of the complete and consistent three-valued logic. This gives us a chance to see how the doxastic operators will function when restricted to this interpretation.

Expansion can be characterized by the set of AGM postulates [5, p. 48-51]. These postulates describe a family of expansion operators. Factually, with the help of the postulates of expansion E1-E5, the operator of expansion can be axiomatically defined. In other words, if the doxastic operator satisfies the postulates of expansion E1-E5, it is an equivalent to the operator expansion of AGM.

We put these expressions as follows: the upper expression is the AGM postulate, and the lower expression is its representation in the logic Ł3. To formulate the corresponding expressions, we relied on the properties of the doxastic operators defined above and on the functioning of truth values. Using truth tables, it is easy to show that all of them are theorems in Ł3.

Closure $K + A$ – belief set **E 1**
$+(A \to B) \to (+A \to +B)$

By adding an implication to the system, we are committed to adding a consequent if an antecedent is added. This is nothing else than the distributivity of the operator with respect to the implication.

Success $A \in (K + A)$ **E 2**
$+A \to A$

It is also a kind of doxastic commitment. If an agent has added a belief to their belief set, they are committed to asserting it.

Inclusion $K \subseteq (K + A)$ **E 3**
$B\& + A \to B$

If we add new sentence A to belief set B we are committed to maintaining B. In other words, we should just add A and reject nothing from the initial belief set.

Vacuity If $A \in K$ then $(K + A) = K$ **E 4**
$A \to (+A \to A)$

It is an interesting case to express because the initial set and an added sentence can be labeled with the same letter. Thus, we have the initial set which implies A. If we add A to this set we should obtain the same set A. In contrast, we can express vacuity in another way explicitly specifying the original set of beliefs: $(B\&A) \to (+A \to B)$.

Monotonicity If $K \subseteq H$ then $(K + A) \subseteq (H + A)$ **E 5**
$(B \to C) \to ((B\& + A) \to (C\& + A))$

Thus, we obtained an admissible interpretation of the AGM postulates for expansion in a three-valued setting. Given that all postulates of the expansion are theorems of the consistent logic Ł3, we can suppose that they are consistent although, indeed, the question of their independence remains open. Despite using the logic Ł3 as a limiter for the semi-formal system of the AGM postulates, here we do not build a sufficiently rigorous formalism to assert that the expressed in this way AGM postulates frame a kind of formal system. Moreover, we suppose that it is not the case. Nevertheless, sufficient expressiveness of the language of the three-valued logic Ł3 and truth tables for the corresponding logical connectives and operators allow us to approach the solution of some issues, such as the purity and primacy of the doxastic operators, the interconnection and mutuality of these operators. Indeed, the expansion operator functions as a normal operator and does not cause any inconvenience being translated into the three-valued logic. Moreover, all its properties were quite predictable. In turn, the contraction operator functions differently; thus, additional formalizations are required to express the AGM postulates of contraction.

Building a contraction operator is considered to be successful if it satisfies the postulates of contraction AGM. However, it was not easy to express these postulates using the operator. To get Ł3 tautologies, we had to strengthen the formulae by expansion operator, but these changes never went against the essence of the postulate.

Closure $K \div A$ – belief set **C 1**
$\div(A \to B) \to (\div B \to A)$

If an implication $(A \to B)$ is indetrmined, i.e. it is deleted from the initial belief set, then contracting by B allows us to save A as an ordinary sentence. Thus, in the reduced set, all its consequences remain, and all other propositions are not affected by the reduction.

Success If $\nvDash A$ then $K \div A \nvDash A$ **C 2**
$\div A \to \neg{+}A$

If the agent contracted their belief set by A, they commit not to add it back, at least not immediately.

Inclusion $K \div A \subseteq K$ **C 3**
$\neg(\div A \to B) \ \to \ \neg{+}B$

The contraction by any proposition shall not cause the expansion of the agent's belief set by propositions that were not contained therein. Any proposition not belonging to the set of beliefs should not appear in it as a result of the contraction by another proposition.

Vacuity If $A \notin K$ then $K \div A = K$ **C 4**
$(B\&\neg{+}A) \to (\div A \to B)$

In contrast, a non-believed sentence does not change the initial set.

Extensionality If $A \Leftrightarrow B$ then $K \div A = K \div B$ **C 5**
$(A \leftrightarrow B) \leftrightarrow (\div A \leftrightarrow \div B)$

These five postulates are already enough for the contraction operator to work normally. If the contraction operator satisfies the rest of five postulates without the postulate of recovery, it is named a withdrawal operator [11, p. 388]. Makinson [11] proved the theorem that the withdrawal operator is sufficient to build the revision function.

5. Trouble with contraction and faces of minimality

The contraction operation is deservedly considered more difficult to implement than the expansion operation. That manifested in our attempt to formulate and prove the postulates of Gärdenfors's contraction. Frequently, we have to use an expansion operator to achieve a valid formula. In addition, this contraction operator only partially satisfies additional postulates for contraction by conjunction. For example, it satisfies *conjunctive inclusion*, which allows us to save one of the conjuncts after contracting by $A\&B$.

Conjunctive inclusion If $K \div (A \wedge B) \nvdash A$ then $K \div (A \wedge B) \subseteq K \div A$
$\neg(\div(A \,\&B) \;\rightarrow A) \;\rightarrow (\div A \;\rightarrow\; \div(A \wedge B))$

However, it partially satisfies the requirement of Conjunctive factoring. If it is necessary to contract a belief set by removing the conjunction of propositions $A\&B$, then there are three ways to do this: to remove A keeping on B; to remove B keeping on A and remove both propositions A and B.

Conjunctive factoring $\div(A\,\&B) \;\rightarrow\; \div A \;\vee\; \div B$

It only works in one direction, while the original postulate requires implications in both directions.

One of the most controversial options for contraction proposed in AGM is the postulate of recovery. We should be able to obtain the initial belief set after adding a belief rejected earlier. Indeed, this is a pretty strict rule.

Recovery $K \subseteq (K \div A) + A$ **C 6**

It is easy to see that we cannot express it in our system in a direct way. In general, of course, we can. We will even get a tautology: $[(B\& \div A)\& + A] \rightarrow B$.

Due to the definition of expansion and contraction operators, their conjunction leads to a contradiction. Thus, we cannot use this way of notation to express the recovery postulate. However, we can formulate and prove a slightly weaker condition.

Weak recovery $(B\& \div A) \rightarrow (+A \rightarrow B)$

One of the most important tasks of the postulate approach is to limit the performance of cognitive actions only to necessary operations. There are several kinds of minimality criteria. The postulate of recovery is one of them. When we change our beliefs, we want to keep on of our old beliefs as much as possible, avoiding unnecessary losses of information and also avoiding unjustified expansion. It is quite difficult to implement the minimality for the doxastic operators. It is often put together from weaker properties like a puzzle.

With the help of contraction and expansion operators, we can formulate very good minimality conditions for the doxastic actions. We can refer to ordinary reasoning or use the AGM postulates of minimality.

Minimality 1 $\div(A \rightarrow B)\& \div B \rightarrow \neg(+A \rightarrow +B)$ **EM 6**

If both the status of the implication and the status of the consequent are undefined, then adding an antecedent to a set of beliefs will not provoke the expansion of the set of beliefs with the consequent. (We need to clarify the status of the consequent because it can be derivable in the system.)

A possible approach to the minimal loss of previous beliefs is the following: sentence B must be discarded in the contraction of K by A only if its presence in the contracted set would lead to A being inferred. This is a part of the minimality for full meet contraction offered by Alchourrón, Gärdenfors and Makinson [1].

Minimality 2 $B\&(\div A \rightarrow \div B) \rightarrow (\div A \rightarrow (B \rightarrow A))$ **CM 2**

Using combinations of these operators, we can formulate various versions of minimality. All of them limit changes to the original belief set by some doxastic commitments.

For example, if there is B in the agent's belief set and an implication $(A \rightarrow B)$ is not true, then contracting the agent's belief set by A should not influence any changes in B.

Minimality 3 $B\&\neg{+}(A \rightarrow B) \;\rightarrow\; (\div A \rightarrow B)$

Indeed, we can obtain many interesting properties with this interpretation. Some of them are unexpected, but others even allow us to refine the properties of doxastic operations. Thus, we can say that our doxastic operators satisfy the basic properties of cognitive actions presented in Gärdenfors's postulates. Moreover, we can express some additional properties of AGM cognitive actions by tautologies of Ł3. It suggests that the way of interpretation which we have chosen is quite acceptable. It allows us to look at the properties of cognitive actions from the other side. Simultaneously, the Ł3 logic itself does not allow us to go beyond the derivability and tautologies of this logic, and that forces us to refine the general properties of the operators based on truth values.

6. Discussion and Conlusion

The considered contraction operator does not reflect the properties of the AGM contraction very well although, whereas it satisfies the necessary postulates. Moreover, it comes from Gärdenfors's requirements for changing the status of beliefs. The operator only partially satisfies the additional postulates, and, to be honest, it demonstrates some strange properties. The operator is quite useful for clarifying the mechanism of reasoning with undetermined propositions. Nevertheless, it covers only part of the cases where the contraction must be applied. Therefore, there is a suspicion that we may consider another candidate for the contraction operator in this system. It can be $\neg{+A}$. It allows you to express the missing properties of the contraction, but it also has its features. Does it make sense to collect the AGM contraction operation from several operators? Can we consider them as separate contraction operators?

Our research allows us to get closer to answering questions of purity and primacy of the doxastic operators and show some connections between them.

1. **The mentioned problem of purity of the doxastic operations** gets a possible solution. Every contraction must be preceded by the expansion. The introduction rule for the contraction operator assumes the execution of the expansion operator. That is quite natural since we usually make decisions to give up a particular belief, based on the acceptance of new information. Generally, propositions with ordinary truth values cannot provoke contraction.
2. **The problem of primacy of the doxastic operations**. The expansion operator turns out to be primary because it handles the epistemic input in the first place. Its presence is required for the contraction operator to be able to work. More frequently, we had to use the extension operator in the formulae expressing Gärdenfors's postulates for the contraction. That was suggested by the truth values of the formulae themselves.
3. **The problem of connection between the doxastic operations**. The contraction operator can be expressed by the expansion and its combination with negation. Given the conditions for its application, we can introduce an individual but not independent introduction rule for it. As we can see, the contraction operator satisfying Gärdenfors's requirements for changing the status of a belief is closely connected with the expansion operator.

Indeed, the three-valued setting is not enough to build a real explanation of the real belief dynamics. Nevertheless, this interpretation is quite useful for clarifying its work with the particular propositions in real situations. It is obvious that the actual process of belief change is very complicated, but it is carried out according to certain rules. The explication of these rules and the search for possible fallacies is a very interesting task. Therefore, we have achieved several goals.

1. Based on the general principles of constructing epistemic systems of Gärdenfors [5], we translated the main concepts and principles of AGM into the setting based on Lukasiewicz's three-valued logic Ł3. We used successfully the strict formalism of complete logic as a limiter for the cognitive operators of AGM and for the rules managing them in belief revision.
2. According to the received interpretation, we obtained new information about the relation between the cognitive actions of expansion and contraction as modal operators in the three-valued logic. We translated the AGM postulates into the language of the three-valued logic and checked the validity of the corresponding expressions with the help of the corresponding truth tables. The values of the received formulae helped us to supplement the interpretation of the postulates of cognitive actions to theorems Ł3. Thus, we received information to clarify the problems of purity, priority, and connection of cognitive operators.
3. As a result of the analysis of relevant theorems, we arrived at a possible formulation of the problem of the minimality of doxastic operators by combining some of their characteristics. Thus, the criterion of minimal changes was revised, and it turned out that it depends not only on the contraction operator.
4. Based on the received features of the interpretation of cognitive actions into a complete logical system with three truth values, which correspond to the cognitive statuses of beliefs in Gärdenfors's epistemic system, we established the possibility of different interpretations of the contraction operator in this system. It is possible since the doxastic modal operators of contraction and expansion correspond to the formulae of the three-valued logic, and they are not external, nor do they require additional axioms. Therefore, we can choose the interpretation of the contraction operator because of how it should change the truth value of the proposition.
5. We have demonstrated the application of translating one formal scheme into another to revise its concepts and rules. We believe that such formal operations make it possible to form a general understanding of the relationships between formal systems, allow us to distinguish the levels of formalization of the system and deepen the concepts of interpretation and models.

We work with modal and other non-classical logics with the master's students in classes. We deal with formal systems and their properties, most of them describe the process of modeling of reasoning at a very high level of idealization, and it is quite difficult for master's students to translate them into practical activity. To be honest, most of them do not perceive topics in a modal or a multi-valued logic as relevant to practice, and they perceive the corresponding tasks as quite interesting but purely formal. On the other hand, belief revision is perceived in a completely different way since it is only partially formalized. It can be a set of guidelines that everyday reasoning shall follow to be considered rational. Such a semi-formal approach leads to the fact that systems based on belief revision are more closely related to practice but do not have the proven characteristics of formal systems. Therefore, they are easy to implement, but there is no guarantee that their requirements are consistent or complete. It is not enough to specify such semi-formal systems only based on practice; they should be coordinated with formal schemes of a higher level.

In fact, interesting methodical results were obtained in the educational task proposed by us. We combined elements of epistemic logic and belief revision and multi-valued logic; we introduced master's students to different levels of formalization and rationality; we showed the method of refining the model not only as a result of its practical implementation but also by matching it with another model; we showed ways of translating a semi-formal system into a formal scheme while working with operators, expressions and various notation methods; we applied semantic tables (truth tables) successfully to prove the adequacy of expressed properties and demonstrated how the truth values affect the expressions, and we received new data on solving some fundamental issues.

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